Sec 1.6 (2) Exact Eqns / Reducible Eqs

An eq. of form $M(x,y)dx + N(x,y)dy = 0$ is "exact"
if $M_y = N_x$ $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$.

★ Can always find an implicit sol. for exact eq.

Idea (Calc 3)

implicit sol $F(x,y) = C$

$\Longrightarrow$ total differential $dF = F_x dx + F_y dy = 0$

$M dx + N dy = 0$

"smooth" functions $F(x,y)$ will have

$(F_x)_y = (F_y)_x$ ("order doesn't matter")

So implicit sol $\Rightarrow$ exact; in fact, exact $\Rightarrow$ implicit sol.

Ex Find an implicit sol for

$$\underbrace{(6xy - y^3)}_{M}dx + \underbrace{(4y + 3x^2 - 3xy^2)}_{N}dy = 0$$

Check if exact : $\left.\begin{array}{l} M_y = 6x - 3y^2 \\ N_x = 6x - 3y^2 \end{array}\right\} \checkmark$

So eq is exact $\Longrightarrow$ exists implicit sol $F(x,y) = C$

Want $F(x,y) = C$ with $\begin{cases} F_x = M = 6xy - y^3 \\ F_y = N = 4y + 3x^2 - 3xy^2 \end{cases}$

Pick either M or N to integrate:

$$F = \int \frac{\partial F}{\partial x} dx = \int M dx = \int (6xy - y^3) dx = 3x^2y - xy^3 + g(y)$$

Need $F_y = N$, $\longrightarrow$ $\underbrace{3x^2 - 3xy^2 + g'(y)}_{F_y} = \underbrace{4y + 3x^2 - 3xy^2}_{N}$

=

so $g'(y) = 4y, \Rightarrow g(y) = 2y^2 + C_1$

Implicit sol $\quad F(x,y) = C$

$$3x^2y - xy^3 + 2y^2 = \boxed{C - C_1}$$

C

Equally valid to pick N:

$$F = \int \frac{\partial F}{\partial y}\, dy = \int N\, dy = \int (4y + 3x^2 - 3xy^2)\, dy$$

$$F = 2y^2 + 3x^2y - xy^3 + h(x)$$

Need $F_x = M \Rightarrow 6xy - y^3 + h'(x) = 6xy - y^3$

$$\Rightarrow h'(x) = 0$$

$$\Rightarrow h(x) = C_1$$

$$F(x,y) = C \longrightarrow 2y^2 + 3x^2y - xy^3 = C$$

☆ Exact eqs might appear differently:

$$\left\{ \frac{dy}{dx} = \frac{-x}{y} \right\} \Rightarrow \begin{array}{l} y\,dy = -x\,dx \\ x\,dx + y\,dy = 0 \end{array}$$

$\begin{array}{l} M = x \\ N = y \end{array} \longrightarrow \left.\begin{array}{l} M_y = 0 \\ N_x = 0 \end{array}\right\} \checkmark$ so eq is exact.

Some 2nd order ODEs can be "reduced" into 1st order

Type "y missing" (but y', y'' okay)

$\longrightarrow$ keep x as indep. var.

$\longrightarrow$ set $\boxed{v = y' = \frac{dy}{dx}}$.

$\longrightarrow$ then $y'' = v'$

$\underline{Ex}$: $\{xy'' + y' = x$, (assume $x>0$)$\}$

$$xv' + v = x \qquad \underline{v = y',\ v' = y''}$$

$$\Longrightarrow \quad v' + \frac{1}{x}v = 1 \qquad (v' + Pv = Q)$$

$$P = \frac{1}{x}, \quad Q = 1$$

$$\rho = e^{\int P(x)dx} = e^{\ln|x|} = x$$

$$\rho \cdot v = \int \rho \cdot Q\, dx + C_1$$

$$xv = \int x\, dx + C_1$$

$$x \cdot v = \frac{1}{2}x^2 + C_1$$

$$y' = \frac{1}{2}x + C_1 \frac{1}{x}$$

(2nd order ODE $\Rightarrow$ 2 unknown const)

$$\Longrightarrow \quad y = \frac{1}{4}x^2 + c_1 \ln|x| + c_2$$

$$\underline{y = \frac{1}{4}x^2 + c_1 \ln x + c_2}$$

<u>Type "x missing"</u> (★ "separable", but $y'' = \frac{d^2y}{dx^2} \cdots$)

$\underline{Ex}$: $y'' = -25y$ $\qquad$ <u>use</u> $\boxed{v = y'} = \frac{dy}{dx}$ again

$$\Longrightarrow y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}$$

$$\boxed{y'' = \frac{dv}{dy} \cdot v}$$

$$v\frac{dv}{dy} = -25y. \quad \text{(separable)}$$

$$\int v\,dv = -25\int y\,dy$$

$$\frac{1}{2}v^2 = -\frac{25}{2}y^2 + C_1$$

$$v = \sqrt{C_1 - 25y^2} = 5\sqrt{C_1 - y^2}$$

$$\frac{dy}{dx} = 5\sqrt{C_1 - y^2} \quad \text{(separable)}$$

$$\int \frac{dy}{\sqrt{C_1 - y^2}} = \int 5 \, dx$$

$$\boxed{\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right) + C}$$

$$\sin^{-1}\left(\frac{y}{\sqrt{C_1}}\right) = 5x + C_2$$

$$\frac{y}{C_1} = \sin(5x + C_2)$$

C_1 came from first integral above

C_2 comes from second

$$\Rightarrow \quad \underline{y = c_1 \sin(5x + c_2)}$$